GREATER BOSTON MATH OLYMPIAD 2006 SOLUTIONS: GRADE 4

Some of the solutions are written using formal mathematical language which could present difficulty for young readers. Adult help in reading this text is recommended.

Problem 1. *In this problem, same letters mean same digits, different letters mean different digits. We have a three-digit number ABB. The product of its digits is a two-digit number AC. The product of the digits of AC is C. What is that original number?*
Answer: 144.
Explanation: since $A \cdot C = C$, either $A = 1$ or $C = 0$. If $C = 0$ then $A \neq 0$ and $A \cdot B \cdot B = 10 \cdot A$, and hence $B \cdot B = 10$, which is impossible. Therefore $A = 1$ and $B \cdot B$ has two digits, the first of which equals 1. Hence $B = 4$.

Problem 2. *A row of 10 digits is written according to the following rule: the first three digits are chosen arbitrarily, and then each next digit is the last digit of the sum of the previous three. For example, starting with 1-2-3 yields 1-2-3-6-1-0-7-8-5-0. Which three digits should go first so that the last three are 1-2-3?*

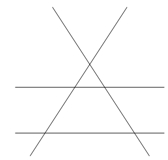
1	**2**	**3**	6	1	0	7	**8**	**5**	**0**

?	**?**	**?**					**1**	**2**	**3**

Answer: 8-9-2.
Explanation: the problem is solved by computing the unknown digits backwards. The only digit X such that the last digit of X+1+2 is 3 is X=0. Hence the fourth digit from the end is 0. In the same way, the only digit Y such that the last digit of Y+0+1 is 2 is Y=1. Hence the fifth digit from the end is 1. Continuing like this shows that the whole row is 8-9-2-9-0-1-0-1-2-3.

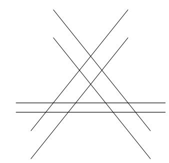
Problem 3. *A book sells for 11 dollars. A customer wants to buy it but only has foreign currency. The exchange rate for the foreign currency is 11 round coins = 15 dollars, 11 square coins = 16 dollars. How many of each coinage should the customer pay?*
Answer: 7 round coins and one square coin.
Explanation: eleven books will sell for 11·11=121 dollars. Since 121=7·15+16, this amount, in dollars, is the same as 77 round coins and 11 square coins, or eleven times a collection of 7 round coins and one square coin. Hence 7 round coins and one square coin equal 11 dollars. This is the only solution, as 11·9 coins is at least 15·9>135 dollars, 11·7 coins is at most 16·7=112 dollars, and replacing a round coin by a square one in a collection of coins increases its value.

Problem 4. *Four lines can be used to draw 2 equilateral triangles on the plane, as shown on the right. What is the maximal number of equilateral triangles which can be drawn using 6 lines? (A triangle is equilateral when all of its angles equal 60°).*



Answer: 8.

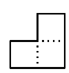
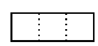
Explanation: eight triangles can be drawn in many ways, one of which is shown on the right. To see that 8 is the maximum, select three of the lines which form an equilateral triangle and let x, y, z be the number of lines drawn to be parallel to these lines. On one hand, $x+y+z \le 6$. On the other hand, the total number of equilateral triangles is not larger than xyz, except for the case $x = y = z = 1$, when it is not larger than 2. Since $xyz \le 8$ when $x, y, z \in \{1, 2, 3\}$ and $x+y+z = 6$, there are not more than 8 equilateral triangles.

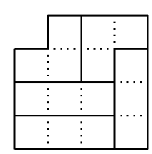


Problem 5. *In Math-annapolis, chicken nuggets can be ordered in boxes of 4,6, and 15. What is the largest number such that you can not order any combination of the above to achieve exactly the number you want?*

Answer: 17.

Explanation: since 17 is an odd number, a box with 15 nuggets has to included in ordering 17 nuggets. However, 15+4>17, hence 17 cannot be ordered exactly. Note that 18=6·3, 19=15+4, 20=4·5, and 21=15+6 can be ordered exactly. Since any number larger than 21 can be obtained by adding a multiple of 4 to one of the numbers 18,19,20,21, every number greater than 17 can be ordered exactly.

Problem 6. *One wants to use tiles of form*  and  *to make a square without a unit size corner (no overlapping of the tiles and no holes are allowed). This can be done when the square is 4-by-4 units, as shown on the right. Among the squares of dimensions 5-by-5, 6-by-6, etc., up to 20-by-20, how many are those for which this can be done?*



Answer: 11.

Explanation: let $Y(x)$ be the figure obtained by cutting a unit square corner from square side length be x units (i.e. $Y(2)$ is one of the tiles, $Y(4)$ is the figure used as an example in the problem formulation). The number of unit squares covering $Y(x)$ equals $x \cdot x - 1$, which is divisibe by three if and only if x is *not* divisible by 3. Since both types of tiles have 3 elements, covering $Y(x)$ is impossible when x is divisible by 3. On the other hand, if $Y(x)$ can be covered then $Y(x+3)$ can be covered as well. Since $Y(2)$ and $Y(4)$ can be covered, $Y(x)$ can be covered whenever $x > 1$ is not divisible by 3. There are 11 such numbers between 5 and 20: 5,7,8,10,11,13,14,16,17,19,20.