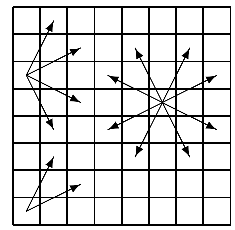
GREATER BOSTON MATH OLYMPIAD 2006 SOLUTIONS: GRADE 6

Some of the solutions are written using formal mathematical language which could present difficulty for young readers. Adult help in reading this text is recommended.

Problem 1. *"Hungry knight" is a piece which can move on the chess board (an 8-by-8 table) according to the following rule: one move is either 2 squares vertically (up or down) and 1 square horizontally (right or left) or 1 square vertically and 2 squares horizontally (see the picture on the right). An "oyster" is another piece which does not move at all. When hungry knight gets to a chess board square on which an oyster stands, it "eats" the poor thing.*



The hungry knight is standing in the lower left hand corner of an empty chess board. Place two oysters at two other squares to maximize the number of moves the hungry knight has to take to eat both of them.

Answer: upper left and lower right corners (requires 11 moves of the hungry knight).

Explanation: the knight can move from one square of the chess board to another in less than 6 moves, except when the two squares are ends of a main diagonal (it takes 6 moves then). This can be verified using the table showing the minimal number of moves from the lower right corner. Since the hungry knight is beginning from a corner, at least one oyster will *not* be across the main diagonal from it. That oyster can always be eaten in less than 6 moves. Then the second oyster can be eaten in 6 moves or less.

5	4	5	4	5	4	5	6
4	3	4	3	4	5	4	5
3	4	3	4	3	4	5	4
2	3	2	3	4	3	4	5
3	2	3	2	3	4	3	4
2	1	4	3	2	3	4	5
3	4	1	2	3	4	3	4
0	3	2	3	2	3	4	5

Hence the maximal total number of moves is not larger than 11, achieved if and only if the oysters are placed across a main diagonal from each other, i.e. in the upper left and lower right corners.

Problem 2. *A row of 10 digits is written according to the following rule: the first three digits are chosen arbitrarily, and then each next digit is the last digit of the sum of the previous three. For example, starting with 1-2-3 yields 1-2-3-6-1-0-7-8-5-0. Which three digits should go first so that the last three are 6-1-7?*

1	**2**	**3**	6	1	0	7	**8**	**5**	**0**

?	**?**	**?**					**6**	**1**	**7**

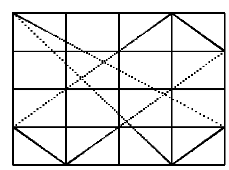
Answer: 7-7-0.

Explanation: the problem is solved by computing the unknown digits backwards. The only digit X such that the last digit of X+6+1 is 7 is X=0. Hence the fourth digit from the end is 0. In the same way, the only digit Y such that the last digit of Y+0+6 is 1 is Y=5. Hence the fifth digit from the end is 5. Continuing like this shows that the whole row is 7-7-0-4-1-5-0-6-1-7.

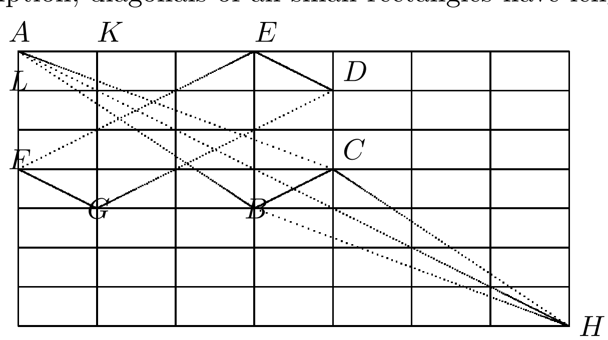
Problem 3. *To pay his income tax, a pirate has to give 9 piles of golden coins, arranged in such a way that no two piles have same number of coins, and no two piles combined*

have same number of coins as a third pile. What is the minimal number of coins the pirate has to pay?
Answer: 81.
Explanation: 81 coins can be put into 9 piles of sizes 1,3,5,7,9,11,13,15 and 17, satisfying the conditions. To prove that it is not possible to have a smaller number of coins, let us call a number a *pile number* if there is a pile with exactly that many coins. Let $x < y$ be the two smallest pile numbers. If $x > 4$ then the second largest pile number is greater than 5, the third gretaer than 6, etc., i.e. at least 5+6+7+...+11+12+13=81≥81 coins total. If $x = 4$, each of the following pairs of numbers has not more than one pile number among them: (5,9), (6,10), (7,11), (8,12), (13,17), (14,18), (15,19), (16,20), hence the total number of coins is not smaller than 4+5+6+7+8+13+14+15+16=88>81. If $x = 3$, same is true for the pairs (4,7), (5,8), (6,9), (10,13), (11,14), (12,15), (16,19), (17,20), hence the total is at least 3+4+5+6+10+11+12+16+17=84>81. If $x = 2$ and $y \neq 3$, partition into pairs (4,6), (5,7), (8,10), (9,11), (12,14), (13,15), (16,18), (17,19) shows that the total is at least 2+4+5+8+9+12+13+16+17=86>81. If $x = 2$ and $y = 3$, 5 is not a pile number, the pair (4,6) has not more than one pile number, and each of the groups (7,8,9,10,11), (12,13,14,15,16) has not more than two pile numbers each, with the total not less than 2+3+4+7+8+12+13+17+18=84>81. If $x = 1$ and $y > 2$ then each of pairs (3,4), (5,6), ... (17,18) has not more than one pile number, and the total is not smaller than 1+3+...+17=81. Finally, if $x = 1$ and $y = 2$ then 3 is not a pile number, and each of the triplets (4,5,6), (7,8,9), ..., (19,20,21) has not more than one pile number, which yields a total of at least 1+2+4+7+...+19+22=94>81.

Problem 4. *A rectangle is divided into 16 equal rectangles, and a parallelogram and a triangle are drawn inside as shown on the right. The perimeter of the parallelogram is 8 feet. The perimeter of the triangle is x feet. Find the largest integer which is smaller than x, and the smallest integer which is larger than x.*



Answer: 8 and 9.
Explanation: consider the drawing below, where A, B, C, D, E, F, G, H, K, L are grid points, and, by assumption, diagonals of all small rectangles have length 1.



Let $|XY|$ be the distance between points X and Y. The triangle inequality says that

$|AB| + |BH| > |AH|$, hence

$$x = |AB| + |AC| + |BC| = |AB| + |BH| + 1 > |AH| + 1 = 8.$$

The triangle inequality also says that $|AB| < |AL| + |LB|$ and $|AC| < |AK| + |KC|$. Since $|AL| < |LK| = 1$ and $|AK| < |LK| = 1$, this implies

$$x = |AB| + |AC| + |BC| < 1 + |LB| + 1 + |KC| + |BC| = 9.$$

Since $8 < x < 9$, the largest integer which is smaller than x equals 8, and the smallest integer which is larger than x equals 9.

Problem 5. *In Math-annapolis, chicken nuggets can be ordered in boxes of 6,9, and 19. What is the largest number such that you can not order any combination of the above to achieve exactly the number you want?*
Answer: 41.
Explanation: since dividing 41 by 3 yields a reminder of 2, two boxes with 19 nuggets have to be included in ordering 41 nuggets. However, 19·2+6>41, hence 41 cannot be ordered exactly. Note that 42=6·7, 43=19+6·4, 44=19·2+6, 45=9·5, 46=19+9·3, and 47=19·2+9 can be ordered exactly. Since any number larger than 47 can be obtained by adding a multiple of 6 to one of the numbers 42,43,44,45,46,47, every number greater than 41 can be ordered exactly.

Problem 6. *A prime number is an integer greater than one divisible only by 1 and itself. A square number is a product of some number and itself. What is the smallest possible difference between a square number and a prime number, if the prime is greater than 3, and the square number is greater than the prime?*
Answer: 2.
Explanation: Since 3·3–7=2, the smallest difference is not larger than 2. However, since $x \cdot x - 1 = (x - 1) \cdot (x + 1)$ is not a prime for $x > 2$, the difference cannot equal 1.