GREATER BOSTON MATH OLYMPIAD 2006 SOLUTIONS: GRADE 7

Some of the solutions are written using formal mathematical language which could present difficulty for young readers. Adult help in reading this text is recommended.

Problem 1. *On a grid paper, there is a rectangle which sides are on the grid. If there are A intersections inside the rectangle and B intersections on its boundary, what's the area of the rectangle?*
Answer: $A + B/2 - 1$.
Explanation: draw a unit square with sides parallel to the grid centered at each of the intersections inside or on the boundary of the rectangle. These squares cover the rectangle without overlapping. The squares centered at the inner points lie completely within the rectangle, and have total area $A \cdot 1 = A$. For the squares centered at the boundary points, exactly half of their area lies within the rectangle, except for the four corner points, for which a quarter of their area lies within the rectangle, for a total are of $A \cdot (1/2) - 4 \cdot (1/4) = A/2 - 1$. Hence the area of the rectangle is $A + B/2 - 1$.

Problem 2. *A row of 10 digits is written according to the following rule: the first three digits are chosen arbitrarily, and then each next digit is the last digit of the sum of the previous three. For example, starting with 1-2-3 yields 1-2-3-6-1-0-7-8-5-0. Which three digits should go first so that the last three are 7-8-1?*

1	**2**	**3**	6	1	0	7	**8**	**5**	**0**

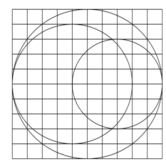
?	**?**	**?**					**7**	**8**	**1**

Answer: 4-7-4.
Explanation: the problem is solved by computing the unknown digits backwards. The only digit X such that the last digit of X+7+8 is 1 is X=6. Hence the fourth digit from the end is 6. In the same way, the only digit Y such that the last digit of Y+6+7 is 8 is Y=5. Hence the fifth digit from the end is 5. Continuing like this shows that the whole row is 4-7-4-5-6-5-6-7-8-1.

Problem 3. *To pay his income tax, a pirate has to give 10 piles of golden coins, arranged in such a way that no two piles have same number of coins, and no two piles combined have same number of coins as a third pile. What is the minimal number of coins the pirate has to pay?*
Answer: 100.
Explanation: 100 coins can be put into 10 piles of sizes 1, 3, 5, 7, 9, 11, 13, 15, 17 and 19, satisfying the conditions. To prove that it is not possible to have a smaller number of coins, let us call a number a *pile number* if there is a pile with exactly that many coins. Let $x < y$ be the two smallest pile numbers. If $x > 5$ then the second largest pile number is greater than 6, the third gretaer than 7, etc., i.e. at least 6+7+…+14+15=105>100 coins total. If $x = 5$, each of the following pairs of numbers has not more than one pile number among them: (6,11), (7,12), (8,13), (9,14), (10,15), (16,21), (17,22), (18,23), hence the total number of coins is not smaller than 5+6+7+8+9+10+16+17+18+19=115>100.

If $x = 4$, each of the following pairs of numbers has not more than one pile number among them: (5,9), (6,10), (7,11), (8,12), (13,17), (14,18), (15,19), (16,20), hence the total number of coins is not smaller than 4+5+6+7+8+13+14+15+16+21=109>100. If $x = 3$, same is true for the pairs (4,7), (5,8), (6,9), (10,13), (11,14), (12,15), (16,19), (17,20), hence the total is at least 3+4+5+6+10+11+12+16+17+18=102>100. If $x = 2$ and $y \neq 3$, partition into pairs (4,6), (5,7), (8,10), (9,11), (12,14), (13,15), (16,18), (17,19) shows that the total is at least 2+4+5+8+9+12+13+16+17+20=106>100. If $x = 2$ and $y = 3$, 5 is not a pile number, the pair (4,6) has not more than one pile number, and each of the groups (7,8,9,10,11), (12,13,14,15,16), etc. has not more than two pile numbers each, with the total not less than 2+3+4+7+8+12+13+17+18+22=106>100. If $x = 1$ and $y > 2$ then each of pairs (3,4), (5,6), ... (17,18), (19,20) has not more than one pile number, and the total is not smaller than 1+3+...+19=100. Finally, if $x = 1$ and $y = 2$ then 3 is not a pile number, and each of the triplets (4,5,6), (7,8,9), etc. has not more than one pile number, which yields a total of at least 1+2+4+7+...+22+25=119>100.

Problem 4. *Three discs are placed within a square as shown on the right. Let x be the area of intersection of two smaller discs. Let y be the area of the part of the larger disc which is not covered by either of the smaller discs. What is the ratio of x and y?*



Answer: 1.
Explanation: since the radii of the discs are $5, 4$ and 3 units, the areas of the three discs are 25π, 16π, and 9π. Since 25=16+9, x equals y, and hence the ratio of x and y equals 1.

Problem 5. *Chicken nuggets can be ordered in boxes of 6,9, and 20. What is the largest number such that you can not order any combination of the above to achieve exactly the number you want?*
Answer: 43.
Explanation: since dividing 43 by 3 yields a reminder of 1, two boxes with 20 nuggets have to be included in ordering 43 nuggets. However, 20·2+6>43, hence 43 cannot be ordered exactly. Note that 44=20+6·4, 45=9·5, 46=20·2+6, 47=20+9·3, 48=6·8, and 49=20·2+9 can be ordered exactly. Since any number larger than 49 can be obtained by adding a multiple of 6 to one of the numbers 44,45,46,47,48, and 49, every number greater than 43 can be ordered exactly.

Problem 6. *What is the last digit of $3^{2007} - 2^{2005}$?*
Answer: 5.
Explanation: by the rules of multiplication, the last digits of $3 \cdot x$ and $2 \cdot x$ are completely determined by the last digit of x. Since the last digit of $3^5 = 243$ is 3, and the last digit of $2^5 = 32$ is 2, the last digits of 3^1, 3^2, ... 3^k repeat themselves with period 4, and the same is true for the last digits of 2^1, 2^2, ... 2^k. Since $2007 = 501 \cdot 4 + 3$, and $2005 = 501 \cdot 4 + 1$, the last digit of 3^{2007} is the same as the last digit of $3^3 = 27$, and the last digit of 2^{2005} is

the same as the last digit of $2^1 = 2$. Hence the last digit of $3^{2007} - 2^{2005}$ is 7–2=5.