GREATER BOSTON MATH OLYMPIAD 2006 SOLUTIONS: GRADE 5

Some of the solutions are written using formal mathematical language which could present difficulty for young readers. Adult help in reading this text is recommended.

Problem 1. *In this multiplication example, same letters mean same digits, different letters mean different digits, and stars mean arbitrary digits. Find A and B.*

$$\begin{array}{r} * * * * * * * \, A \\ A \\ \hline BBBBBBBBB \end{array}$$

Answer: $A = 9$, $B = 1$.

Explanation: inequality $B < A$ must be satisfied, otherwise the result of dividing number $BBBBBBBBB$ by A would have nine digits. Since B is last digit of a $A \cdot A$, the following three pairs (A, B) could be possible: $A = 6, B = 4$; $A = 8, B = 4$, and $A = 9, B = 1$. The first two cases are impossible because then $BBBBBBBBB$ is not divisible by A. Hence the multiplication example is $111111111 = 9 \cdot 12345679$.

Problem 2. *A row of 10 digits is written according to the following rule: the first three digits are chosen arbitrarily, and then each next digit is the last digit of the sum of the previous three. For example, starting with 1-2-3 yields 1-2-3-6-1-0-7-8-5-0. Which three digits should go first so that the last three are 7-7-7?*

1	**2**	**3**	6	1	0	7	**8**	**5**	**0**

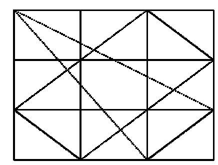
?	?	?					**7**	**7**	**7**

Answer: 1-7-1.
Explanation: the problem is solved by computing the unknown digits backwards. The only digit X such that the last digit of X+7+7 is 7 is X=3. Hence the fourth digit from the end is 3. In the same way, the only digit Y such that the last digit of Y+3+7 is 7 is Y=7. Hence the fifth digit from the end is 7. Continuing like this shows that the whole row is 1-7-1-9-7-7-3-7-7-7.

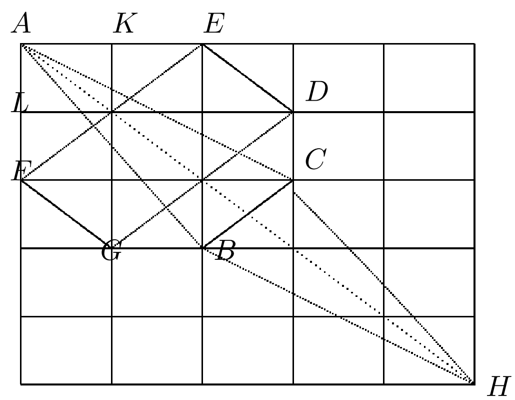
Problem 3. *To pay his income tax, a pirate has to give 8 piles of golden coins, arranged in such a way that no two piles have same number of coins, and no two piles combined have same number of coins as a third pile. What is the minimal number of coins the pirate has to pay?*
Answer: 64.
Explanation: 64 coins can be put into 8 piles of sizes 1,3,5,7,9,11,13, and 15, satisfying the conditions. To prove that it is not possible to have a smaller number of coins, let us call a number a *pile number* if there is a pile with exactly that many coins. Let $x < y$ be the two smallest pile numbers. If $x > 4$ then the second largest pile number is greater than 5, the third gretaer than 6, etc., i.e. at least 5+6+7+. . . +11+12=68>64 coins total. If $x = 4$, each of the following pairs of numbers has not more than one pile

number among them: (5,9), (6,10), (7,11), (8,12), (13,17), (14,18), (15,19), hence the total number of coins is not smaller than 4+5+6+7+8+13+14+15=72>64. If $x = 3$, same is true for the pairs (4,7), (5,8), (6,9), (10,13), (11,14), (12,15), (16,19), etc., hence the total is at least 3+4+5+6+10+11+12+16=67>64. If $x = 2$ and $y \neq 3$, partition into pairs (4,6), (5,7), (8,10), (9,11), (12,14), (13,15), (16,18), etc. shows that the total is at least 2+4+5+8+9+12+13+16=69>64. If $x = 2$ and $y = 3$, 5 is not a pile number, the pair (4,6) has not more than one pile number, and each of the groups (7,8,9,10,11), (12,13,14,15,16) has not more than two pile numbers each, with the total not less than 2+3+4+7+8+12+13+17=66>64. If $x = 1$ and $y > 2$ then each of pairs (3,4), (5,6), ..., (15,16) has not more than one pile number, and the total is not smaller than 1+3+...+15=64. Finally, if $x = 1$ and $y = 2$ then 3 is not a pile number, and each of the triplets (4,5,6), (7,8,9), ..., (19,20,21) has not more than one pile number, which yields a total of at least 1+2+4+7+10+13+16+19=72>64.

Problem 4. *A rectangle is divided into 9 equal rectangles, and a parallelogram and a triangle are drawn inside as shown on the right. The perimeter of the parallelogram is 6 feet. The perimeter of the triangle is x feet. Find the largest integer which is smaller than x, and the smallest integer which is larger than x.*



Answer: 6 and 7.
Explanation: consider the drawing below, where A, B, C, D, E, F, G, H, K, L are grid points, and, by assumption, diagonals of all small rectangles have length 1.



Let $|XY|$ be the distance between points X and Y. The triangle inequality says that $|AB| + |BH| > |AH|$, hence

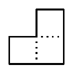
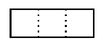
$$x = |AB| + |AC| + |BC| = |AB| + |BH| + 1 > |AH| + 1 = 6.$$

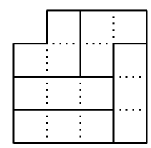
The triangle inequality also says that $|AB| < |AL| + |LB|$ and $|AC| < |AK| + |KC|$. Since $|AL| < |LK| = 1$ and $|AK| < |LK| = 1$, this implies

$$x = |AB| + |AC| + |BC| < 1 + |LB| + 1 + |KC| + |BC| = 7.$$

Since $6 < x < 7$, the largest integer which is smaller than x equals 6, and the smallest integer which is larger than x equals 7.

Problem 5. *In Math-annapolis, chicken nuggets can be ordered in boxes of 4,6, and 17. What is the largest number such that you can not order any combination of the above to achieve exactly the number you want?*
Answer: 19.
Explanation: since 19 is an odd number, a box with 17 nuggets has to be included in ordering 19 nuggets. However, 17+4>19, hence 19 cannot be ordered exactly. Note that 20=4·5, 21=17+4, 22=6·3+4, and 23=17+6 can be ordered exactly. Since any number larger than 23 can be obtained by adding a multiple of 4 to one of the numbers 20, 21, 22, 23, every number greater than 19 can be ordered exactly.

Problem 6. *One wants to use tiles of form*  *and*  *to make a square without a unit size corner. This can be done when the square is 4-by-4 units, as shown on the right. Among the squares of dimensions 5-by-5, 6-by-6, 7-by-7, etc., up to 100-by-100, how many are those for which this can be done?*



Answer: 11.
Explanation: let $Y(x)$ be the figure obtained by cutting a unit square corner from square side length be x units (i.e. $Y(2)$ is one of the tiles, $Y(4)$ is the figure used as an example in the problem formulation). The number of unit squares covering $Y(x)$ equals $x \cdot x - 1$, which is divisibe by three if and only if x is *not* divisible by 3. Since both types of tiles have 3 elements, covering $Y(x)$ is impossible when x is divisible by 3. On the other hand, if $Y(x)$ can be covered then $Y(x+3)$ can be covered as well. Since $Y(2)$ and $Y(4)$ can be covered, $Y(x)$ can be covered whenever $x > 1$ is not divisible by 3. There are 64 such numbers between 5 and 100: 5,7,8,10,11...,97,98,100.